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Graded contractions of affine Lie algebras

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Abstract. The method of graded contractions, based on the preservation of automorphisms of finite order, is applied to the affine Lie algebras and their representations, to yield a new class of infinite-dimensional Lie algebras and representations. After the introduction of the horizontal and vertical gradings, and the algorithm to find the horizontal toroidal gradings, some general properties of the graded contractions are discussed and compared with the Inönü–Wigner contractions. The example of $\hat{A}_2$ is discussed in detail.

1. Introduction

In this paper, I describe the *graded contractions* of general affine Lie algebras and their representations, and illustrate the method with $\hat{A}_2$. Generally speaking, contractions of Lie algebras are deformations, or singular transformations, of the constants of structure. They were introduced in physics by Inönü and Wigner [1] in order to provide a formal relationship between the kinematical groups of Einstein's special relativity and Galilean relativity. In general, contractions are interesting because they relate, in a meaningful way, different Lie algebras such that various properties of the contracted (or limit) algebra can be obtained from the initial algebra. This is particularly promising in the study of non-semisimple Lie algebras (which often can be seen as the outcome of a contraction), because their representation theory and their general structure are not as elegant and uniform as the semisimple Lie algebras.

Although the Lie algebras most familiar to physicists are *simple*, such as $su(2)$, $su(3)$ or E_6, many algebras of physical interest are likely to be non-semisimple. The situation is similar with the infinite-dimensional Lie algebras. Since the Kac–Moody and Virasoro algebras represent a rather restricted class of algebras, their contractions lead to a totally new class of infinite-dimensional Lie algebras, which might well be relevant in physics. An example of an infinite-dimensional algebra which can be seen as a contraction of a Kac–Moody algebra is the oscillator (or Heisenberg) algebra, with commutation relations:

$$[\overline{h}, a_n] = 0 \qquad [a_m, a_n] = m\delta_{m+n,0}\overline{h}.$$

(It is shown below equation (3.6) that this algebra is a rather trivial graded contraction.) Since the early work of Inönü and Wigner, the method has been generalized in many directions (some of which are given in [2–7]) and applied to various problems in physics (see, for instance, [8–15]). To my knowledge, the first systematical treatment of Inönü–Wigner contractions of Kac–Moody algebras appeared in [16].

Recently, non-semisimple affine Lie algebras have been used in string theory, in the context of Wess–Zumino–Witten (WZW) models [17], where they occur in the expression of the current algebras of the models. String backgrounds based on non-semisimple WZW models have been constructed in [18], and a general class of exact conformal field theories, with integral Virasoro central charges, have been constructed in [19]. Other constructions appear in [20]. Although these models have been the main motivation for the present work, it has interest of its own and is not restricted to these applications.

In this paper, I apply a method of contraction [22, 23] based on the preservation of a *grading* of a Lie algebra: a decompositon of the algebra into eigenspaces of an automorphism of finite order. I describe the algorithm to find the grading preserving contractions, or *graded contractions*, of an affine Lie algebra. Starting with an affine algebra we obtain different (i.e. non-isomorphic) infinite-dimensional Lie algebras. The interest of this particular method is when we require, for some physical reasons, one or many automorphisms (e.g. parity or time reversal) to be admitted by the limit algebras. The systematical study of Lie gradings has been initiated in [21], as a powerful tool in Lie theory. When it comes to non-semisimple algebras, the graded contractions could be most useful in studying the gradings. Indeed, in some cases, a contraction is the only way to build representations of such 'exotic' infinite algebras, whose representation theory is yet to be understood. To summarize, many properties of non-semisimple Lie algebras might be obtained from a contraction, and if, for some physical reason, an automorphism of finite order is preserved throughout that contraction, then the formalism of graded contractions is possibly more appropriate. An advantage of this method is that it applies simultaneously to all the algebras and representations which admit a common grading.

The method described in [22, 23] is implicitly applicable to infinite-dimensional Lie algebras, but it is studied systematically (for all affine Lie algebras) for the first time here. The particular case $\hat{A}_1$ has been considered in [25]. I also introduce the concept of *vertical* and *horizontal* gradings, which do not exist with the finite-dimensional algebras. Having in mind the purpose of applying these results to high-energy physics, I consider also the contractions of (integrable irreducible highest weight) representations, and discuss various properties of the contracted algebras. Although the present formalism can be applied to superalgebras, I do not consider them here.

I close this section by reviewing briefly the method of graded contractions, introduced in [22, 23] and summarized in [24].

1.1. Definition of graded contractions

A *grading* of a (finite- or infinite-dimensional) Lie algebra $\mathfrak{g}$ is a vector decomposition:

$$\mathfrak{g} = \bigoplus_{\mu\in\Gamma} \mathfrak{g}_\mu \qquad \text{such that } [\mathfrak{g}_\mu, \mathfrak{g}_\nu] \subseteq \mathfrak{g}_{\mu+\nu} \tag{1.1}$$

where μ and ν belong to an Abelian finite grading group Γ. I follow the notation of [22–24].

Along with the decomposition (1.1), a grading of a $\mathfrak{g}$-module V is a splitting:

$$V = \bigoplus_{\mu\in\Gamma} V_\mu \qquad \text{with } \mathfrak{g}_\mu \cdot V_\nu \subseteq V_{\mu+\nu}. \tag{1.2}$$

As mentioned previously, a grading is associated with an automorphism of finite order (which may follow from physical restrictions). For instance, in [20] one may notice that, in the construction of a WZW model, if it is possible to split the initial group into the 'coset' part and the 'subgroup' part by the matrix that rotates the generators, then it follows that if this matrix is an automorphism, then the contracted algebra admits this automorphism as well.

The *graded contractions* of $\mathfrak{g}$ are defined by introducing parameters $\varepsilon_{\mu,\nu}$ (real or complex, according to the field underlying the algebra), such that the contracted algebra $\mathfrak{g}^{\varepsilon}$ has the same vector space as the initial algebra $\mathfrak{g}$, but modified commutation relations:

$$[\mathfrak{g}_\mu, \mathfrak{g}_\nu]_\varepsilon \equiv \varepsilon_{\mu,\nu}[\mathfrak{g}_\mu, \mathfrak{g}_\nu] \subseteq \varepsilon_{\mu,\nu}\mathfrak{g}_{\mu+\nu}. \tag{1.3}$$

Similarly, the graded contractions of representations are defined through the introduction of parameters $\psi_{\mu,\nu}$, which deform the action of $\mathfrak{g}$ on V such that they preserve the grading (1.2):

$$\mathfrak{g}_\mu \cdot^{\psi} V_\nu \equiv \psi_{\mu,\nu}\mathfrak{g}_\mu \cdot V_\nu \subseteq \psi_{\mu,\nu} V_{\mu+\nu}. \tag{1.4}$$

The contraction parameters ε and ψ must satisfy the equations

$$\varepsilon_{\mu,\nu}\varepsilon_{\mu+\nu,\sigma} = \varepsilon_{\nu,\sigma}\varepsilon_{\sigma+\nu,\mu} = \varepsilon_{\mu,\sigma}\varepsilon_{\mu+\sigma,\nu} \tag{1.5}$$

and

$$\varepsilon_{\mu,\nu}\psi_{\mu+\nu,\sigma} = \psi_{\nu,\sigma}\psi_{\mu,\nu+\sigma} = \psi_{\mu,\sigma}\psi_{\nu,\mu+\sigma}. \tag{1.6}$$

The solutions of these two sets of equations, substituted back into (1.3) and (1.4), provide the contractions of the algebra $\mathfrak{g}$ and its representations. To each set of parameters ε (which defines a contracted algebra), the corresponding solutions of (1.6) for the ψ's yield contractions of the representation. (More details and remarks are given in [22–24]. The contraction of the tensor product of the representations has been introduced in [23].)

In the next section, I present the concepts of vertical and horizontal gradings of affine Lie algebras, and illustrate them with $\hat{A}_2$. In section 3, I discuss the graded contractions and some properties. The purpose of this paper is not to provide huge (and not particularly useful) tables of contracted algebras. Instead, one can rely on the program presented in [24], given a specific grading or algebra. I rather describe gradings of affine algebras and some general features of the contractions, emphasizing the main differences with the traditional methods.

2. Gradings of affine Lie algebras

Consider a simple complex Lie algebra $\mathfrak{g}$, and the corresponding affine untwisted Kac–Moody algebra $\hat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}k$, where $\mathbb{C}[t, t^{-1}]$ is the associative algebra of the Laurent polynomials in t, and k is a central extension. The first term in the sum, $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, is called the *loop algebra* of $\mathfrak{g}$. Given $a, b \in \mathfrak{g}$, the commutation relations in $\hat{\mathfrak{g}}$ read

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + mkB(a, b)\delta_{m+n,0} \tag{2.1}$$

where $m, n \in \mathbb{Z}$, $[a, b]$ is the commutator in $\mathfrak{g}$, and $B(a, b)$ is the Killing form of $\mathfrak{g}$. General properties of infinite-dimensional Lie algebras can be found in [26–29] and references therein.

Here, I distinguish two classes of gradings: *horizontal* and *vertical*. The first is a grading of the finite algebra $\mathfrak{g}$ that is preserved through the affinization process. The vertical gradings are *not* inherent in $\mathfrak{g}$, but are rather given by the gradings of $\mathbb{C}[t, t^{-1}]$. Obviously, these two types of gradings can be combined to provide gradings which are neither vertical nor horizontal. One can think of these two types of gradings as being the building blocks of the gradings of $\hat{\mathfrak{g}}$. Vertical gradings have no analogue in the finite-dimensional algebras.

2.1. Horizontal (toroidal) gradings

As mentioned in the previous paragraph, any grading of the *finite* Lie algebra $\mathfrak{g}$ can be 'elevated' to a grading of the affine Lie algebra $\hat{\mathfrak{g}}$, and it is then called 'horizontal grading'. There exists no uniform prescription to find all the gradings of a general Lie algebra. A comprehensive list of gradings exists only for the simple Lie algebras of rank two and some of rank three. However, I sketch here a method which provides an important class of gradings of semisimple Lie algebras: the *toroidal gradings*. The gradings of a *finite* Lie algebra $\mathfrak{g}$ are associated with automorphisms of finite order (or conjugacy classes of elements of finite order (EFO)) of the corresponding compact Lie group K [30, 31]. (An elementary introduction to the EFO theory, sufficient for our purposes, is given in [33].) Kac's theory of EFO provides a prescription to identify the conjugacy classes of EFO, and hence the gradings. Such as described below, the action of an EFO leads to a *toroidal* $\mathbb{Z}_N$ grading (for an EFO of order N), and one can use it to grade simultaneously a Lie algebra and its irreducible representations. (It is called 'toroidal' because it is a coarsening of the *toroidal*—or *Cartan*—decomposition.) This method provides, in a straightforward way, the unique diagonal representative of a conjugacy class of EFO in any irreducible representation of $\mathfrak{g}$. All one needs to know is the weight system of the representation.

To grade an algebra, one must consider the *adjoint* representation, for which the weight system is the root system of $\mathfrak{g}$. If $\mathfrak{g}$ has rank r, then the EFO is represented by an array of non-negative integers $[s_0, \ldots, s_r]$, with 1 as the greatest common divisor. To each root α—and therefore each basis element of $\mathfrak{g}$—is associated an eigenvalue

$$\exp\frac{2\pi \mathrm{i}}{M}\langle\alpha, \boldsymbol{s}\rangle \tag{2.2}$$

where $\langle\alpha, \boldsymbol{s}\rangle = \sum_{j=1}^{r} a_j s_j$, if $\alpha = \sum_{j=1}^{r} a_j \alpha_j$ (α_j are the simple roots of $\mathfrak{g}$). Also, $M = s_0 + \sum_{j=1}^{r} c_j s_j$, the c_j being the components (called *marks*) of the highest root $\Psi = \sum_{j=1}^{r} c_j \alpha_j$ of $\mathfrak{g}$. The vector of marks is annihilated by the affine Cartan matrix $\mathbf{A}$ (i.e. $\sum_{k=0}^{r} c_j A_{jk} = 0$, for all k). The elements e_α belong to the eigenvalue given by (2.2), for any positive or negative root α, and all the elements of the Cartan subalgebra belong to the eigenvalue 1. The *order* of the EFO is $N = MC$, where C is given in table 6 of [31] for all the simple Lie algebras. The grading group is then $\Gamma = \mathbb{Z}_N$.

This can be generalized to any weight system. Let $V(\Lambda)$ be an irreducible $\mathfrak{g}$-module with highest weight Λ, that can be Cartan-decomposed as

$$V(\Lambda) = \bigoplus_{\lambda\in\Omega(\Lambda)} V_\lambda^\Lambda \qquad V_\lambda^\Lambda = \{v \in V(\Lambda)|\ hv = \lambda(h)v\} \tag{2.3}$$

where h is an element of $\mathfrak{h}$, the Cartan subalgebra of $\mathfrak{g}$, and $\Omega(\Lambda)$ is the weight system of the module. To each weight $\lambda \in \Omega(\Lambda)$ a grading decomposition (1.1) is obtained by determining eigenvalues similar to (2.2),

$$v_\lambda \longrightarrow v_\lambda \exp\frac{2\pi \mathrm{i}}{M}\langle\lambda, \boldsymbol{s}\rangle \tag{2.4}$$

where $\langle\lambda, \boldsymbol{s}\rangle = \sum_{j=1}^{n} b_j s_j$, if $\lambda = \sum_{j=1}^{n} b_j \alpha_j$. The value of M is the same as in (2.2). Obviously, there are other—*non-toroidal*—gradings of $\mathfrak{g}$ which can serve as horizontal gradings, but I do not consider them hereafter. In fact, they are often related to toroidal gradings (for example, a grading of $\mathfrak{g}$ can be provided by an EFO of a larger group, which contains $\mathfrak{g}$). A comprehensive classification of all such gradings does not yet exist.

A general $\mathbb{Z}_{m1} \otimes \cdots \otimes \mathbb{Z}_{mk}$ is obtained by 'mixing' gradings $\mathbb{Z}_{m1}, \mathbb{Z}_{m2}, \ldots$ found by using the EFO. If each $\mathbb{Z}_{mj}$ provides a decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \bigoplus_{\mu_j \in \mathbb{Z}_{m_j}} \mathfrak{g}_{\mu_j} \tag{2.5}$$

for $j = 1, \ldots, k$, then a $\Gamma = \mathbb{Z}_{m1} \otimes \cdots \otimes \mathbb{Z}_{mk}$ grading is obtained as follows,

$$\mathfrak{g} = \bigoplus_{\mu \in \Gamma} \mathfrak{g}_{\mu = (\mu_1 \ldots \mu_k)} \tag{2.6}$$

where $\mathfrak{g}_{(\mu_1 \ldots \mu_k)} = \mathfrak{g}_{\mu_1} \cap \cdots \cap \mathfrak{g}_{\mu_k}$. An example is given at the end of section 2.3.

Once we have a Γ grading (1.1) of a *finite* Lie algebra $\mathfrak{g}$ (obtained from the EFO, or otherwise), then its affine Lie algebra $\hat{\mathfrak{g}}$ admits the horizontal grading

$$\hat{\mathfrak{g}} = \bigoplus_{\mu \in \Gamma} \hat{\mathfrak{g}}_{\mu} \tag{2.7}$$

where

$$\hat{\mathfrak{g}}_0 = (\mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}k \qquad \hat{\mathfrak{g}}_{\mu \neq 0} = \mathfrak{g}_{\mu} \otimes \mathbb{C}[t, t^{-1}]. \tag{2.8}$$

(The identity element of Γ is denoted by 0.) Note that, whether $\hat{\mathfrak{g}}$ is contracted or not, each $\hat{\mathfrak{g}}_{\mu \neq 0}$ carries a representation space for the subalgebra $\hat{\mathfrak{g}}_0$.

In order to find gradings of an irreducible integrable highest weight representation of $\hat{\mathfrak{g}}$, it is useful to express the gradings of $\hat{\mathfrak{g}}$ in terms of the root vectors of $\hat{\mathfrak{g}}$. If $\alpha \in \Delta_{\mathfrak{g}}$ is a root in $\mathfrak{g}$ (with components $\alpha_1, \ldots, \alpha_r$), then the element $e_\alpha \otimes t^m$ can be denoted by $E_{\alpha + m\delta}$. The root α_0 is given by $\delta = \alpha_0 + \Psi$ (Ψ is the highest root of $\mathfrak{g}$). Therefore the root vector can be expressed solely in terms of the affine simple roots, $(\alpha_0, \alpha_1, \ldots, \alpha_r)$, or in terms of δ and the finite simple roots, $(\delta, \alpha_1, \ldots, \alpha_r)$. To any element h_k in the Cartan subalgebra of $\mathfrak{g}$ is associated the root vector $E^k_{m\delta}$. In the case of a $\mathbb{Z}_N$ grading provided by (2.2), $E_{\alpha+m\delta}$ belongs to the subspace $\hat{\mathfrak{g}}_\mu$, for any m, if e_α belongs to the grading subspace $\mathfrak{g}_\mu$. Obviously, all the vectors $E_{m\delta}$ belong to $\hat{\mathfrak{g}}_0$, as do the elements of the Cartan subalgebra of $\hat{\mathfrak{g}}$.

Finally, I repeat that the gradings of a finite algebra are not always manifestly related to an EFO and that there are other types of gradings (e.g. the generalized Pauli matrices used in [21]). Such gradings are the result of an *outer* automorphism of $\mathfrak{g}$, whereas the EFO correspond to *inner* automorphisms. In any event, once a grading (2.6) of a finite algebra is known, the equations (2.7)–(2.8) provide the corresponding horizontal grading of the affine algebra.

2.2. Vertical gradings

Similarly, the vertical $\mathbb{Z}_N$ gradings are given by the action of a root of the unity, $\exp(2\pi \mathrm{i}/N)$, on the associative algebra $\mathbb{C}[t, t^{-1}]$,

$$\phi : t \to \exp\left(\frac{2\pi \mathrm{i}}{N}\right) t \tag{2.9}$$

such that the element t^m belongs to the eigenvalue $\exp((2\pi \mathrm{i}/N)m)$, and we write $t^m \in \mathbb{C}[t, t^{-1}]_{m \bmod N}$. Therefore, the grading can be written

$$\mathbb{C}[t, t^{-1}] = \bigoplus_{j=0}^{N-1} \mathbb{C}[t, t^{-1}]_j \tag{2.10}$$

where

$$\mathbb{C}[t, t^{-1}]_j = \oplus \mathbb{C}\ t^{j+kN} \qquad k \in \mathbb{Z}. \tag{2.11}$$

Accordingly, the grading of the Kac–Moody algebra $\hat{\mathfrak{g}}$ is

$$\hat{\mathfrak{g}} = \bigoplus_{j\in\mathbb{Z}_N} \hat{\mathfrak{g}}_j \tag{2.12}$$

where

$$\hat{\mathfrak{g}}_0 = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]_0) \oplus \mathbb{C}k \qquad \hat{\mathfrak{g}}_j = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]_j. \tag{2.8$'$}$$

For example, a $\mathbb{Z}_3$ grading is

$$\hat{\mathfrak{g}}_0 = \mathbb{C}k + \cdots + \mathfrak{g} \otimes t^{-3} + \mathfrak{g} \otimes t^0 + \mathfrak{g} \otimes t^3 + \cdots$$
$$\hat{\mathfrak{g}}_1 = \cdots + \mathfrak{g} \otimes t^{-2} + \mathfrak{g} \otimes t^1 + \mathfrak{g} \otimes t^4 + \cdots$$
$$\hat{\mathfrak{g}}_2 = \cdots + \mathfrak{g} \otimes t^{-1} + \mathfrak{g} \otimes t^2 + \mathfrak{g} \otimes t^5 + \cdots.$$

In terms of the affine root vectors discussed below (2.8), every element $E_{\alpha+m\delta}$ belongs to the subspace $\hat{\mathfrak{g}}_{m \bmod N}$.

2.3. An example: $\hat{A}_2$

A general element of the rank two, eight-dimensional simple Lie algebra A_2 (or $sl(3, \mathbb{C})$) can be written in the matrix form:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -(a+e) \end{pmatrix}. \tag{2.13}$$

Upon affinization, this algebra becomes the infinite-dimensional Lie algebra $\hat{A}_2 = (A_2 \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}k$, where k is a central element. However, the usual matrix product must be modified so as to satisfy the commutation relations (2.1).

The *horizontal* gradings of $\hat{A}_2$ are provided by an EFO $\boldsymbol{s} = [s_0, s_1, s_2]$ which describes a conjugacy class of elements of order $N = MC$, where $M = s_0 + s_1 + s_2$ and $C = 3/(gcd(3; s_1+2s_2))$ (see [33]). The only element of order two is given by $\boldsymbol{s} = [0, 1, 1]$ and provides, according to (2.2), the grading

$$\begin{aligned} (A_2)_0 &\equiv \mathfrak{h} + \mathbb{C}e_{\pm(\alpha_1+\alpha_2)} \\ (A_2)_1 &\equiv \mathbb{C}e_{\pm\alpha_1} + \mathbb{C}e_{\pm\alpha_2}. \end{aligned} \tag{2.14}$$

$\mathfrak{h} = \mathbb{C}h_1 \dot{+} \mathbb{C}h_2$ is the Cartan subalgebra of A_2. In terms of the affine root vectors, the corresponding grading subspaces of $\hat{\mathfrak{g}}$ are generated by

$$\begin{aligned} (\hat{A}_2)_0 &= \{E_{m\delta}, E_{\pm(\alpha_1+\alpha_2)+m\delta}, k\} \\ (\hat{A}_2)_1 &= \{E_{\pm\alpha_1+m\delta}, E_{\pm\alpha_2+m\delta}\} \qquad m \in \mathbb{Z}. \end{aligned} \tag{2.14$'$}$$

Using $\delta = \alpha_0 + \Psi_{A_2} = \alpha_0 + \alpha_1 + \alpha_2$, we can write, for instance, $E_{\alpha_1+m\delta}$ as $E_{m\alpha_0+(m+1)\alpha_1+m\alpha_2}$, etc.

The finite algebra A_2 admits two elements of order three, $[1, 1, 1]$ and $[0, 1, 0]$, which correspond to

$$\begin{aligned} (A_2)_0 &\equiv \mathfrak{h} \\ (A_2)_1 &\equiv \mathbb{C}e_{\alpha_1} + \mathbb{C}e_{\alpha_2} + \mathbb{C}e_{-(\alpha_1+\alpha_2)} \\ (A_2)_2 &\equiv \mathbb{C}e_{-\alpha_1} + \mathbb{C}e_{-\alpha_2} + \mathbb{C}e_{\alpha_1+\alpha_2} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned}(A_2)_0 &\equiv \mathfrak{h} + \mathbb{C}e_{\pm\alpha_2} \\ (A_2)_1 &\equiv \mathbb{C}e_{\alpha_1} + \mathbb{C}e_{\alpha_1+\alpha_2} \\ (A_2)_2 &\equiv \mathbb{C}e_{-\alpha_1} + \mathbb{C}e_{-(\alpha_1+\alpha_2)}\end{aligned} \tag{2.16}$$

respectively. Therefore, the $\mathbb{Z}_3$ grading of $\hat{A}_2$ given by [1, 1, 1] is

$$\begin{aligned}(\hat{A}_2)_0 &= \{E_{m\delta}, k\} \\ (\hat{A}_2)_1 &= \{E_{\alpha_1+m\delta}, E_{\alpha_2+m\delta}, E_{-\alpha_1-\alpha_2+m\delta}\} \\ (\hat{A}_2)_2 &= \{E_{-\alpha_1+m\delta}, E_{-\alpha_2+m\delta}, E_{\alpha_1+\alpha_2+m\delta}\} \qquad m \in \mathbb{Z}\end{aligned} \tag{2.15$'$}$$

and the grading [0, 1, 0] is

$$\begin{aligned}(\hat{A}_2)_0 &= \{E_{m\delta}, E_{\pm\alpha_2+m\delta}, k\} \\ (\hat{A}_2)_1 &= \{E_{\alpha_1+m\delta}, E_{\alpha_1+\alpha_2+m\delta}\} \\ (\hat{A}_2)_2 &= \{E_{-\alpha_1+m\delta}, E_{-(\alpha_1+\alpha_2)+m\delta}\} \qquad m \in \mathbb{Z}.\end{aligned} \tag{2.16$'$}$$

These expressions illustrate the fact that, for an horizontal grading, if $e_\alpha \in \mathfrak{g}_\mu$, then $E_{\alpha+m\delta} \in \hat{\mathfrak{g}}_\mu$, for all $m \in \mathbb{Z}$.

From section 2.2, the *vertical* $\mathbb{Z}_2$ grading is given by

$$\begin{aligned}(\hat{A}_2)_0 &\equiv (A_2 \otimes t^{2m}) \oplus \mathbb{C}k = \{E_{\alpha+2m\delta}, k\} \\ (\hat{A}_2)_1 &\equiv A_2 \otimes t^{2m+1} = \{E_{\alpha+(2m+1)\delta}\}\end{aligned} \tag{2.17}$$

the $\mathbb{Z}_3$ grading by

$$\begin{aligned}(\hat{A}_2)_0 &\equiv (A_2 \otimes t^{3m}) \oplus \mathbb{C}k = \{E_{\alpha+3m\delta}, k\} \\ (\hat{A}_2)_1 &\equiv A_2 \otimes t^{3m+1} = \{E_{\alpha+(3m+1)\delta}\} \\ (\hat{A}_2)_2 &\equiv A_2 \otimes t^{3m+2} = \{E_{\alpha+(3m+2)\delta}\}\end{aligned} \tag{2.18}$$

and a general $\mathbb{Z}_N$ grading by

$$\begin{aligned}(\hat{A}_2)_0 &\equiv (A_2 \otimes t^{Nm}) \oplus \mathbb{C}k = \{E_{\alpha+Nm\delta}, k\} \\ (\hat{A}_2)_1 &\equiv A_2 \otimes t^{Nm+1} = \{E_{\alpha+(Nm+1)\delta}\} \\ (\hat{A}_2)_2 &\equiv A_2 \otimes t^{Nm+2} = \{E_{\alpha+(Nm+2)\delta}\} \\ &\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\ (\hat{A}_2)_{N-1} &\equiv A_2 \otimes t^{Nm+N-1} = \{E_{\alpha+(Nm+N-1)\delta}\}.\end{aligned} \tag{2.19}$$

To illustrate the meaning of expression (2.6), I display below a mixed vertical–horizontal grading. If we mix the horizontal decomposition (2.14$'$) with the vertical grading (2.17), we get the following $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ grading:

$$\begin{aligned}(\hat{A}_2)_{00} &= \{E_{2m\delta}, E_{\pm(\alpha_1+\alpha_2)+2m\delta}, k\} \\ (\hat{A}_2)_{01} &= \{E_{(2m+1)\delta}, E_{\pm(\alpha_1+\alpha_2)+(2m+1)\delta}\} \\ (\hat{A}_2)_{10} &= \{E_{\pm\alpha_1+2m\delta}, E_{\pm\alpha_2+2m\delta}\} \\ (\hat{A}_2)_{11} &= \{E_{\pm\alpha_1+(2m+1)\delta}, E_{\pm\alpha_2+(2m+1)\delta}\} \qquad m \in \mathbb{Z}.\end{aligned} \tag{2.20}$$

The first $\mathbb{Z}_2$ index corresponds to the grading (2.14$'$), and the second index to the decomposition (2.17). One can verify that (2.20) satisfies the relation (1.1).

2.4. Gradings of representations

In this section, I describe and give some examples of the gradings of integrable irreducible highest weight representations of untwisted affine Lie algebras, given a vertical or horizontal grading of the algebra.

An irreducible highest weight integrable module $V(\Lambda)$ of $\hat{\mathfrak{g}}$ is labelled by its *highest weight* $\Lambda = (n; \Lambda_0, \ldots, \Lambda_r)$, where $n, \Lambda_0, \ldots, \Lambda_r$ are non-negative integers. As the finite case (see (2.3)), it can be weight decomposed as

$$V(\Lambda) = \bigoplus_{\lambda \in \hat{\mathfrak{h}}^*} V_\lambda^\Lambda \tag{2.21}$$

where the weight $\lambda = (n; \lambda_0, \ldots, \lambda_r)$ has multiplicity $m_\lambda^\Lambda = \dim V_\lambda^\Lambda$. $\hat{\mathfrak{h}}$ is the Cartan subalgebra of $\hat{\mathfrak{g}}$. An invariant of $V(\Lambda)$ is the *level* $\Lambda(k) = \sum_{j=0}^r \check{c}_j \Lambda_j$ (k, central element of $\hat{\mathfrak{g}}$), where the $\check{c}_j$ are the *comarks* of $\hat{\mathfrak{g}}$, defined by $\sum_{j=0}^r A_{jk} \check{c}_k = 0$, for all j ($\mathbf{A}$: affine Cartan matrix of $\hat{\mathfrak{g}}$). The integer n in λ is called the *null depth*, and is equal to the number of α_0's that must be subtracted from Λ to reach λ. The null depth determines the vertical gradings.

In order to grade the module $V(\Lambda)$ compatibly with some given grading of $\hat{\mathfrak{g}}$, we consider the action of root vectors E_α on the vectors in $V(\Lambda)$ so as to coarsen the Cartan decomposition (2.21). To achieve this, we first express the roots in the basis of fundamental weights $\omega_0, \ldots, \omega_r$:

$$\begin{aligned}
\alpha_0 &= (1; \alpha_0^0, \ldots, \alpha_0^r) = \delta + \alpha_0^0 \omega_0 + \cdots + \alpha_0^r \omega_r \\
\alpha_1 &= (0; \alpha_1^0, \ldots, \alpha_1^r) = \alpha_1^0 \omega_0 + \cdots + \alpha_1^r \omega_r \\
\vdots \quad & \quad \vdots \\
\alpha_r &= (0; \alpha_r^0, \ldots, \alpha_r^r) = \alpha_r^0 \omega_0 + \cdots + \alpha_r^r \omega_r
\end{aligned} \tag{2.22}$$

where the coefficients are given by the affine Cartan matrix, $\alpha_k^j = A_{jk}$, for $j, k = 0, \ldots, r$, and $\delta = (1; 0, \ldots, 0)$. In the case of $\hat{A}_2$, the Cartan matrix is

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{2.23}$$

so that

$$\alpha_0 = (1; 2, -1, -1) \qquad \alpha_1 = (0; -1, 2, -1) \qquad \alpha_2 = (0; -1, -1, 2). \tag{2.24}$$

We use E_α, expressed in the ω-basis, and the fact that $E_\alpha \cdot V_\lambda^\Lambda \subseteq V_{\lambda+\alpha}^\Lambda$ in order to find a compatible grading of $V(\Lambda)$. For instance, the horizontal $\mathbb{Z}_2$ grading of $\hat{A}_2$ given by (2.14′) reads, in this basis,

$$\begin{aligned}
(\hat{A}_2)_0 &= \{(m; 0, 0, 0), (m; \mp 2, \pm 1, \pm 1), k\} \\
(\hat{A}_2)_1 &= \{(m; \mp 1, \pm 2, \mp 1), (m; \mp 1, \mp 1, \pm 2)\} \qquad m \in \mathbb{Z}.
\end{aligned} \tag{2.25}$$

In this simple case, we can find by inspection that

$$V_0 = \{(m; 2\mathbb{Z} + 1, \mathbb{Z}, \mathbb{Z}\} \qquad V_1 = \{(m; 2\mathbb{Z}, \mathbb{Z}, \mathbb{Z}\} \tag{2.26}$$

together with (2.25) satisfies (1.2).

To illustrate the $\mathbb{Z}_3$ grading, I consider the irrep $\Lambda = (1, 0, 0)$. Down to null depth $n = 10$, its weight space decomposition has the form:

λ :	w_0	w_1	w_2	w_3	w_4	w_5
$n = 0$	1					
$n = 1$	2	1				
$n = 2$	5	2				
$n = 3$	10	5	1			
$n = 4$	20	10	2	1		
$n = 5$	36	20	5	2		
$n = 6$	65	36	10	5		
$n = 7$	110	65	20	10	1	
$n = 8$	185	110	36	20	2	
$n = 9$	360	185	65	36	5	1
$n = 10$	481	360	110	65	10	2

where

$$
\begin{aligned}
w_0 &= (1, 0, 0)\\
w_1 &= (-1, 1, 1), (0, -1, 2), (2, -2, 1), (3, -1, -1)\\
w_2 &= (-2, 0, 3), (1, -3, 3), (4, 0, -3)\\
w_3 &= (-3, 2, 2), (-1, -2, 4), (3, 2, -4), (5, -2, -2)\\
w_4 &= (-4, 1, 4), (-3, -1, 5), (5, -5, 1), (0, 5, -4), (2, 4, -5), (6, -1, -4)\\
w_5 &= (-5, 3, 3), (-2, -3, 6), (4, -6, 3), (7, -3, -3)
\end{aligned}
$$

and the vertical strings contain the weight multiplicities, given in [29].

The $\mathbb{Z}_3$ grading $[1, 1, 1]$ of (2.15′) can be read

$$
\begin{aligned}
(\hat{A}_2)_0 &= \{(m; 0, 0, 0), k\}\\
(\hat{A}_2)_1 &= \{(m; -1, 2, -1), (m; -1, -1, 2), (m; 2, -1, -1)\}\\
(\hat{A}_2)_2 &= \{(m; 1, -2, 1), (m; 1, 1, -2), (m; -2, 1, 1)\}.
\end{aligned}
\tag{2.27}
$$

To find the corresponding grading of $V(\Lambda)$, one can choose the highest weight $(0; 1, 0, 0)$ to belong to V_0, and act iteratively on this weight with the elements of various grading subspaces (2.27). From (1.2), one finds, for all the weights down to $n = 10$,

$$
\begin{aligned}
V_0 = \{&(1, 0, 0), (-2, 0, 3), (1, 3, -3), (4, 0, -3), (-5, 3, 3), (-2, 6, -3), (7, -3, -3),\\
&(4, 3, -6), \ldots\}\\
V_1 = \{&(0, 2, -1), (3, -1, -1), (-3, 2, 2), (3, 2, -4), (-3, 5, -1), (0, 5, -4),\\
&(6, -1, -4), \ldots\}\\
V_2 = \{&(-1, 1, 1), (2, 1, -2), (-1, 4, -2), (-4, 1, 4), (5, -2, -2), (2, 4, -5),\\
&(5, 1, -5), \ldots\}
\end{aligned}
\tag{2.28}
$$

plus all the permutations of the last two components λ_1 and λ_2 of each weight (e.g. $(4, -3, 0) \in V_0$). The null depth n is omitted in the weight because the grading does not depend on it. The grading has been chosen so that the weight $(1, 0, 0)$ belongs to V_0. The straightforward way to obtain (2.28) is by using (1.2) and applying all the elements of the different grading subspaces ($\hat{\mathfrak{g}}_1$ and $\hat{\mathfrak{g}}_2$ from (2.27)) on V_0 so as to find a subset of each grading subspace of $V(\Lambda)$. Proceeding iteratively, we then apply the same elements (through (1.2)) on the identified elements of the V_μ found in the first step, to find further elements of V_μ. The grading (2.28) lies in the direction of the *principal slicing* [29] of the weight system.

For the grading [0, 1, 0] of (2.16′) we have

$$\begin{aligned}(\hat{A}_2)_0 &= \{(m;0,0,0),(m;\pm1,\pm1,\mp2),k\}\\ (\hat{A}_2)_1 &= \{(m;-1,2,-1),(m;-2,1,1)\}\\ (\hat{A}_2)_2 &= \{(m;1,-2,1),(m;2,-1,-1)\}.\end{aligned} \tag{2.29}$$

By proceeding as for (2.28), we find the decomposition:

$$\begin{aligned}V_0 = \{&(1,0,0),(0,-1,2),(2,1,-2),(-1,-2,4),(3,2,-4),(-4,4,1),\\ &(-3,5,-1),(5,-5,1),(6,-4,-1),(-5,3,3),(-2,-3,6),(-2,6,-3),\\ &(4,3,-6),(4,-6,3),(7,-3,-3),\ldots\}\\ V_1 = \{&(-1,1,1),(0,2,-1),(-2,0,3),(1,3,-3),(4,-3,0),(3,-4,2),\\ &(5,-2,-2),(-3,-1,5),(2,4,-5),(2,-5,4),(6,-1,-4),\ldots\}\\ V_2 = \{&(2,-2,1),(3,-1,-1),(-2,3,0),(-3,2,2),(1,-3,3),(-1,4,-2),\\ &(4,0,-3),(-4,1,4),(0,-4,5),(0,5,-4),(5,1,-5),\ldots\}.\end{aligned} \tag{2.30}$$

Again, the grading is independent of the null depth.

From (2.19), we see that a general vertical $\mathbb{Z}_N$ grading has the form

$$\begin{aligned}(\hat{A}_2)_0 &= \{(Nm,\alpha)\}\\ (\hat{A}_2)_1 &= \{(Nm+1,\alpha)\}\\ &\vdots\\ (\hat{A}_2)_k &= \{(Nm+k,\alpha)\}\end{aligned} \tag{2.31}$$

where $m \in \mathbb{Z}$, and α is any root of $\hat{A}_2$. Now the grading depends on the null depth only. The corresponding grading of $V(\Lambda)$ is

$$V_k = \{(k,\alpha)\} \qquad k = 0,\ldots,N-1 (\mathrm{mod}\, N) \qquad \text{for all } \alpha. \tag{2.32}$$

3. Graded contractions

In section 1, I have defined the graded contractions of any Lie algebra and its representations. In section 2, I have described the horizontal and vertical gradings, and, more particularly, the toroidal gradings of affine Lie algebras and their irreducible representations. These are the basic elements needed to contract an algebra and its representations. It is now straightforward to obtain the graded contractions of affine Lie algebras, which form a new class of infinite-dimensional Lie algebras. To summarize the contraction of algebras: one gets an horizontal grading (2.8) by using the expression (2.2) to find the eigenspaces of the EFO, or a vertical grading by using (2.11) and (2.8′). To find the graded contractions, one just replaces the solutions of (1.5) in the modified commutation relations (1.3). The grading of representations has been described and illustrated in section 2.4.

The purpose of this section is *not* to display huge lists of contractions, but rather to describe their general properties. There exists a computer program [24] that provides the solutions of equations (1.5) and (1.6), given the grading group Γ and the structure of the grading (i.e. generic or non-generic). Each solution then provides a contraction of the algebra or the representation.

The most straightforward definition of graded contractions of an affine Lie algebra is (after (1.3)) to deform the commutator (2.1) as

$$[a \otimes t^m, b \otimes t^n]_\varepsilon \equiv \varepsilon_{\mu,\nu}[a \otimes t^m, b \otimes t^n] \\ = \varepsilon_{\mu,\nu}[a, b] \otimes t^{m+n} + \varepsilon_{\mu,\nu} mkB(a, b)\delta_{m+n,0} \tag{3.1}$$

where $a \otimes t^m \in \mathfrak{g}_\mu, b \otimes t^n \in \mathfrak{g}_\nu$, a vertical or horizontal grading. (As discussed below, an interesting alternative is to deform simultaneously the Killing form B [32], so that it becomes possible to preserve the central extension whereas the first term in (3.1) is put to zero.)

3.1. Comparison with Inönü–Wigner contractions

First, we compare the Inönü–Wigner contraction of a Kac–Moody algebra (studied in [16]) with the particular case of a $\mathbb{Z}_2$ graded contraction. We write the basis of the algebra $\hat{\mathfrak{g}}$ as T_m^a, where $a = 1, \ldots, \dim \hat{\mathfrak{g}}$ and $m \in \mathbb{Z}$, and the commutation relations

$$[T_m^a, T_n^b] = \mathrm{i} f_c^{a,b} T_{m+n}^c + \tfrac{1}{2} km\delta^{a,b}\delta_{m+n}. \tag{3.2}$$

Then, we decompose $\hat{\mathfrak{g}}$ following Inönü–Wigner, by writing the underlying vector space as $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_0 + \hat{\mathfrak{g}}_1$, where $\hat{\mathfrak{g}}_0 = \{T_m^\alpha\}, \alpha = 1, 2, \ldots, r$, forms a subalgebra of $\hat{\mathfrak{g}}$, and $\hat{\mathfrak{g}}_1 = \{T_m^i\}, i = r + 1, r + 2, \ldots, \dim \hat{\mathfrak{g}}$ is its complementary subspace. The commutation relations (3.2) must take the form

$$\begin{aligned} &[T_m^\alpha, T_n^\beta] = \mathrm{i} f_\gamma^{\alpha,\beta} T_{m+n}^\gamma + \tfrac{1}{2} km\delta^{\alpha,\beta}\delta_{m+n,0} \\ &[T_m^\alpha, T_n^i] = \mathrm{i} f_j^{\alpha,i} T_{m+n}^j \\ &[T_m^i, T_n^j] = \mathrm{i} f_\alpha^{i,j} T_{m+n}^\alpha + \tfrac{1}{2} km\delta^{i,j}\delta_{m+n,0} \end{aligned} \tag{3.3}$$

in order to define an Inönü–Wigner contraction. The contraction is then defined by multiplying all the basis elements of the vector subspace $\hat{\mathfrak{g}}_1$ by a contraction parameter ε so that, in the limit $\varepsilon \to 0$, the commutators in the third row of (3.3) vanish. Thus, the resulting algebra admits a $\mathbb{Z}_2$ grading, where $\hat{\mathfrak{g}}_0 = \{T_m^\alpha, k\}$ and $\hat{\mathfrak{g}}_1 = \{T_n^i\}$. Therefore, one can say that the Inönü–Wigner contraction of an affine algebra is a particular case of a $\mathbb{Z}_2$ graded contraction, with $\varepsilon_{0,0} = 1 = \varepsilon_{0,1}$ and $\varepsilon_{1,1} = 0$. Obviously, there are other graded contractions which lead to an Inönü–Wigner contraction.

As mentioned in [25] for the particular case of $\hat{A}_1$, the contractions of an algebra $\hat{\mathfrak{g}}$ include semidirect sums of the initial $\hat{\mathfrak{g}}$ with an infinite-dimensional Abelian ideal, or 'translation' algebra. In other words, among the possible contractions of $\hat{\mathfrak{g}}$, one finds the algebra $\hat{\mathfrak{g}} \triangleright \mathfrak{a}$, where $\mathfrak{a}$ is an infinite-dimensional Abelian ideal of the contracted algebra. This may be surprising because it is specific to the infinite-dimensional Lie algebras, and cannot occur in the finite cases. This occurs with a *vertical* grading, where, from (2.8), $\hat{\mathfrak{g}}_0 = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]_0) \oplus \mathbb{C}k \approx \hat{\mathfrak{g}}$. If $\varepsilon_{0,\mu} = 1$ for all μ, and all the other parameters vanish, then the subalgebra $\hat{\mathfrak{g}}$ is preserved, and so are the commutators which involve this subalgebra and the remaining basis elements. Since all the remaining commutators vanish, the corresponding ideal is Abelian.

In fact, the graded contractions allow one to go much further than this. Whenever $\varepsilon_{0,0} = 1$, the subalgebra $\hat{\mathfrak{g}}$ (i.e. the initial algebra) will be contained in the contracted algebra, either in direct or semidirect sums. I will illustrate this with $\mathbb{Z}_2$ contractions. In addition to the contraction mentioned previously, there are two other non-trivial contractions, namely one where $\varepsilon_{0,0} = 1, \varepsilon_{0,1} = 0 = \varepsilon_{1,1}$, and the other with $\varepsilon_{0,0} = 0 = \varepsilon_{0,1}, \varepsilon_{1,1} = 1$. For the first, only the subalgebra $\hat{\mathfrak{g}}$ is preserved, so that the contracted algebra is $\hat{\mathfrak{g}}^\varepsilon = \hat{\mathfrak{g}} \oplus \mathfrak{a}$, where

$\mathfrak{a}$ is Abelian (and, obviously, infinite). Under this contraction the vector space underlying $\mathfrak{a}$ no longer carries a representation space of the subalgebra $\hat{\mathfrak{g}}$ in the adjoint representation. In the second case, the only commutation relations that are not deformed to zero are $[\hat{\mathfrak{g}}_1, \hat{\mathfrak{g}}_1]$. Therefore, the subspace $\hat{\mathfrak{g}}_0$ becomes Abelian, and the commutation relations involving any of its elements also vanish.

We note also that the centre is modified under a contraction, as in the finite case. Whereas the centre of an affine algebra consists only of its central extension, it usually becomes bigger after a contraction. For instance, in the first $\mathbb{Z}_2$ contraction displayed in the previous paragraph the centre also includes all the elements of the subspace $\hat{\mathfrak{g}}_1$. In the second $\mathbb{Z}_2$ contraction, the centre includes the elements of the subalgebra $\hat{\mathfrak{g}}_0$ (i.e. the initial algebra). Depending on the particular grading that is preserved, and depending on the initial algebra, there might be additional elements in the centre.

3.2. Generators of positive root vectors

Another interesting property that is modified under a contraction is the minimal set of generators of positive root vectors. In general, the greater the number of contracted commutators (i.e. zero after contraction), the greater is the set of such generators. Below, I illustrate this point by discussing in detail some examples with $\hat{A}_1$ and $\hat{A}_2$.

The set of positive root vectors of $\hat{A}_1$ is given by $E_{p\alpha_0+q\alpha_1}$, where p and q are positive integers such that $-1 \leqslant p - q \leqslant 1$. The gradings of $\hat{A}_1$ are easy to visualize if we write these vectors as

$$\begin{array}{llllll} E_{\alpha_1} & E_{\alpha_1+\delta} & E_{\alpha_1+2\delta} & E_{\alpha_1+3\delta} & E_{\alpha_1+4\delta} & \cdots \\ & E_{\delta} & E_{2\delta} & E_{3\delta} & E_{4\delta} & \cdots \\ & E_{-\alpha_1+\delta} & E_{-\alpha_1+2\delta} & E_{-\alpha_1+3\delta} & E_{-\alpha_1+4\delta} & \cdots . \end{array}$$

Now I will show explicitly how to obtain the generators for all the $\mathbb{Z}_2$ contractions. (This was done in [25] but I obtain them here in a more systematic way, which is easier to generalize to other algebras and gradings.)

For the *horizontal* grading, $\hat{\mathfrak{g}}_0 = \{E_{m\delta}, k;\ m \geqslant 1\}$ are the elements of the second row, and $\hat{\mathfrak{g}}_1 = \{E_{\alpha}, E_{\alpha+m\delta}, E_{-\alpha+m\delta};\ m \geqslant 1\}$ corresponds to the first and third rows. To find the generators of positive root vectors for the contraction $\varepsilon_{0,0} = 1 = \varepsilon_{0,1}, \varepsilon_{1,1} = 0$, we consider each element of the array, one at the time, and see if it can be obtained by commutation of previous generators, by taking into account the contraction parameters. It is convenient to start from the left, and from the bottom to the top, so the two elements that we keep first are E_{α_1} and $E_{-\alpha_1+\delta}$. The next two elements, E_{δ} and $E_{\alpha_1+\delta}$, can be obtained by the commutator of the first two, so we do not need them as generators. The next (and the last) one to be retained is $E_{-\alpha_1+\delta}$, which cannot be obtained from any commutator of the other elements. Therefore, the set of positive roots of the graded contractions is generated by three vectors: E_{α_1}, $E_{-\alpha_1+\delta}$, and $E_{-\alpha_1+2\delta}$.

By a similar reasoning, we find that the generators corresponding to the contraction $\varepsilon_{0,0} = 1, \varepsilon_{0,1} = 0 = \varepsilon_{1,1}$ are

$$E_{\alpha_1}, \quad E_{\alpha_1+(2k+1)\delta}, \quad E_{-\alpha_1+2\delta}, \quad E_{(2k+1)\delta}, \quad E_{-\alpha_1+(2k+1)\delta} \qquad k \geqslant 0.$$

For the contraction $\varepsilon_{0,0} = 0 = \varepsilon_{0,1}$ and $\varepsilon_{1,1} = 1$, the generators are

$$E_{\alpha_1}, \quad E_{\alpha_1+(2k+1)\delta}, \quad E_{(2k+1)\delta}, \quad E_{-\alpha_1+(2k+1)\delta} \qquad k \geqslant 0.$$

The *vertical* grading, for which $\hat{\mathfrak{g}}_0 = \{E_{\alpha_1}, E_{\alpha_1+2m\delta}, E_{2m\delta}, E_{-\alpha_1+2m\delta}, k;\ m \geqslant 1\}$ and $\hat{\mathfrak{g}}_1 = \{E_{\alpha_1+(2m-1)\delta}, E_{(2m-1)\delta}, E_{-\alpha_1+(2m-1)\delta};\ m \geqslant 1\}$ consists of the elements of the odd columns for $\hat{\mathfrak{g}}_0$, and the even columns for $\hat{\mathfrak{g}}_1$. For this grading (which is non-generic, because $[\hat{\mathfrak{g}}_0, \hat{\mathfrak{g}}_0] = 0$), we have two contractions $\begin{pmatrix} * & \varepsilon_{0,1} \\ \varepsilon_{0,1} & \varepsilon_{1,1} \end{pmatrix}$, together with their respective generators of positive root vectors:

$$\begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{with generators } E_{\alpha_1}, E_{(k+1)\delta}, E_{-\alpha_1+\delta}$$
$$\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with generators } E_{\alpha_1+k\delta}, E_{-\alpha_1+(k+1)\delta} \qquad k \geqslant 0.$$

I now illustrate this with the algebra $\hat{A}_2$, and its $\mathbb{Z}_2$ gradings (2.14) and (2.17). As before, it is convenient to display the positive root vectors of $\hat{A}_2$ in an array:

$$\begin{array}{llllll}
E_{\alpha_1+\alpha_2} & E_{\alpha_1+\alpha_2+\delta} & E_{\alpha_1+\alpha_2+2\delta} & E_{\alpha_1+\alpha_2+3\delta} & E_{\alpha_1+\alpha_2+4\delta} & \cdots \\
E_{\alpha_2} & E_{\alpha_2+\delta} & E_{\alpha_2+2\delta} & E_{\alpha_2+3\delta} & E_{\alpha_2+4\delta} & \cdots \\
E_{\alpha_1} & E_{\alpha_1+\delta} & E_{\alpha_1+2\delta} & E_{\alpha_1+3\delta} & E_{\alpha_1+4\delta} & \cdots \\
 & E_{\delta} & E_{2\delta} & E_{3\delta} & E_{4\delta} & \cdots \\
 & E_{-\alpha_1+\delta} & E_{-\alpha_1+2\delta} & E_{-\alpha_1+3\delta} & E_{-\alpha_1+4\delta} & \cdots \\
 & E_{-\alpha_2+\delta} & E_{-\alpha_2+2\delta} & E_{-\alpha_2+3\delta} & E_{-\alpha_2+4\delta} & \cdots \\
 & E_{-(\alpha_1+\alpha_2)+\delta} & E_{-(\alpha_1+\alpha_2)+2\delta} & E_{-(\alpha_1+\alpha_2)+3\delta} & E_{-(\alpha_1+\alpha_2)+4\delta} & \cdots .
\end{array}$$

The *horizontal* grading provided by the EFO $[0, 1, 1]$ (see (2.14′)) consists in the top, middle and bottom rows for $\hat{\mathfrak{g}}_0$, and the four remaining rows for $\hat{\mathfrak{g}}_1$. It is a generic grading for which the generators are

$$E_{\alpha_1},\ E_{\alpha_2},\ E_{\alpha_1+\alpha_2}, E_{-(\alpha_1+\alpha_2)+\delta}$$

for the contraction $\begin{pmatrix} \varepsilon_{0,0} & \varepsilon_{0,1} \\ \varepsilon_{0,1} & \varepsilon_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$,

$$E_{\alpha_1}, \quad E_{\alpha_2}, \quad E_{\alpha_1+\alpha_2}, \quad E_{-(\alpha_1+\alpha_2)+\delta} \quad E_{\alpha_1+k\delta}, \quad E_{\alpha_2+k\delta}, \quad E_{-\alpha_1+k\delta}, \quad E_{-\alpha_2+k\delta} \qquad k \geqslant 1$$

for the contraction $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and

$$E_{\alpha_1}, \quad E_{\alpha_2}, \quad E_{-(\alpha_1+\alpha_2)+\delta}, \quad E_{\alpha_1+k\delta}, \quad E_{\alpha_2+k\delta} \quad E_{-\alpha_1+k\delta}, \quad E_{-\alpha_2+k\delta} \qquad k \geqslant 1$$

for the contraction $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

The *vertical* $\mathbb{Z}_2$ grading (2.17) has the elements of $\hat{\mathfrak{g}}_0$ given by the odd columns, and the elements of $\hat{\mathfrak{g}}_1$ given by the even columns. It is another generic grading for which the generators are

$$E_{\alpha_1},\ E_{\alpha_2},\ E_{-(\alpha_1+\alpha_2)+\delta},\ E_{\delta},\ E_{2\delta},\ E_{-(\alpha_1+\alpha_2)+2\delta}$$

for the contraction $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$,

$$E_{\alpha_1}, \quad E_{\alpha_2}, \quad E_{-(\alpha_1+\alpha_2)+2\delta}, \quad E_{\Delta+(2k-1)\delta} \qquad k \geqslant 1$$

where Δ represents all the roots (including the zero root) of $\mathfrak{g}$, for the contraction $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $E_{\Delta+2k\delta}$, $(k \geqslant 1)$, for the contraction $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

3.3. Contractions involving a deformation of the bilinear form

As mentioned at the beginning of this section, one can define the contracted commutators by allowing the invariant bilinear form B to be contracted as well [32]. Given an horizontal grading, with $a \in \mathfrak{g}_\mu$ and $b \in \mathfrak{g}_\nu$, the commutator (3.1) is then modified to

$$[a \otimes t^m, b \otimes t^n]_\varepsilon = \varepsilon_{\mu,\nu}[a, b] \otimes t^{m+n} + \varepsilon_{\mu,\nu}\gamma_{\mu,\nu} mkB(a, b)\delta_{m+n,0} \tag{3.4}$$

where B is replaced by $B^\gamma = \gamma B$. This permits the preservation of the second term on the right-hand side, even if $\varepsilon = 0$, by choosing γ such that $\gamma\varepsilon$ is constant. From [32], $B^\gamma(\mathfrak{g}_\mu, \mathfrak{g}_\nu) \equiv \gamma_{\mu,\nu} B(\mathfrak{g}_\mu, \mathfrak{g}_\nu)$, where γ must satisfy [32]

$$\varepsilon_{\mu,\nu}\gamma_{\mu+\nu,\sigma} = \varepsilon_{\nu,\sigma}\gamma_{\mu,\nu+\sigma} \qquad \gamma_{\mu,\nu} = \gamma_{\nu,\mu}. \tag{3.5}$$

To illustrate this, consider the $\mathbb{Z}_2$ contraction: $\varepsilon_{0,0} = 1, \varepsilon_{0,1} = 0 = \varepsilon_{1,1}$. The corresponding solutions of (3.5) for γ are $\gamma_{0,1} = 0$, and $\gamma_{0,0}, \gamma_{1,1}$ free. One can choose $\gamma_{1,1}$ to approach infinity as $\varepsilon_{1,1}$ approaches zero, such that $\varepsilon_{1,1}\gamma_{1,1} = K$, a constant. The commutators of the contracted algebra then become

$$\begin{aligned} &[(a \otimes t^m)_0, (b \otimes t^n)_0]_\varepsilon = [a, b] \otimes t^{m+n} + \gamma_{0,0} mkB(a, b)\delta_{m+n,0} \\ &[(a \otimes t^m)_0, (b \otimes t^n)_1]_\varepsilon = 0 \\ &[(a \otimes t^m)_1, (b \otimes t^n)_1]_\varepsilon = KmkB(a, b)\delta_{m+n,0}. \end{aligned} \tag{3.6}$$

As mentioned at the very beginning of this paper, the oscillator algebra can be obtained through the trivial $\mathbb{Z}_2$ contraction $\varepsilon_{0,0} = \varepsilon_{0,1} = \varepsilon_{1,1} = 0$ (for which the three parameters γ are free) with $\gamma_{0,1} = 0 = \gamma_{1,1}$ and $\gamma_{0,0}$ approaching infinity as $\varepsilon_{0,0}$ approaches zero, so that $\varepsilon_{0,0}\gamma_{0,0} = K$. The central term above is then preserved. Actually, when performed this way the oscillator algebra is a subalgebra of the initial affine algebra. To get the true oscillator algebra, one just takes the trivial grading $\hat{\mathfrak{g}}_0 = \hat{\mathfrak{g}}, \hat{\mathfrak{g}}_1 = 0$, which shows that even this important algebra is a rather trivial graded contraction.

4. Conclusion and outlook

A natural continuation of this work consists of applying the previous results in theoretical physics. In general, any physical system or theory which is a limit of another system may have its Lie algebra related to the algebra of the initial system by some contraction procedure. This point might serve to identify potential contractions of affine Lie algebras (e.g. in conformal field theory). Much work remains to be done on the mathematical applications of the present paper. For instance, we have not considered the contractions of *twisted* affine algebras, although the graded contractions can be used in this context.

Let me finish by mentioning an interesting aspect of the contraction of affine Lie algebras which I plan to consider in more detail in the near future: the behaviour, under contractions, of the *extended* affine algebra $\hat{\mathfrak{g}}^{\rm e} = \hat{\mathfrak{g}} \lhd \mathfrak{V}$, where $\mathfrak{V}$ is the Virasoro algebra associated with $\hat{\mathfrak{g}}$. From the structure of the semidirect sum, we can see that, given a *vertical* $\mathbb{Z}_N$ grading (2.8), the $\mathbb{Z}_N$ grading of $\hat{\mathfrak{g}}^{\rm e}$ is

$$\begin{aligned} &\hat{\mathfrak{g}}^{\rm e}_0 = (\mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}k \oplus \mathbb{C}L_{0 \bmod N} \\ &\hat{\mathfrak{g}}^{\rm e}_{j\neq 0} = \mathfrak{g}_j \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}L_{j \bmod N} \qquad j = 1, \ldots, N-1. \end{aligned}$$

On the other hand, in the case of an horizontal grading, the Virasoro algebra $\mathfrak{V}$ is contained completely in the grading subspace $\hat{\mathfrak{g}}^{\rm e}_0$.

It would be very interesting to study the representations of $\mathfrak{V}$ obtained through the Sugawara construction, and see if the conclusions above are compatible with this construction. The basis elements L_n of $\mathfrak{V}$ are defined by

$$L_n = \tfrac{1}{2} \sum_{m\in\mathbb{Z}} : \sum_{i,j=1}^{\dim \mathfrak{g}} (a_i \otimes t^{-m})(a_j \otimes t^{m+n}) : B(a_i, a_j)$$

where $B(\cdot, \cdot)$ is the Killing form of $\mathfrak{g}$, which is just the Kronecker symbol in the bases usually utilized in the Sugawara construction. However, because a general grading of $\mathfrak{g}$ is not always associated with such a basis, we must keep it explicitly in the sum.

Next one would have to examine each step of this construction, by taking into account the contraction parameters introduced both in the commutation relations (1.3), the action on the representation (1.4) and the bilinear form (see [32]). Each term in the sum then takes the form

$$\sigma^{\psi}(: \mathfrak{g}_\mu \mathfrak{g}_\nu :)V_\rho = \varepsilon_{\mu,\nu}\psi_{\mu+\nu,\rho}\gamma_{\mu,\nu}\sigma(: \mathfrak{g}_\mu \mathfrak{g}_\nu :)V_\rho.$$

The detailed investigation of this construction is beyond the scope of the present paper. I plan to study the contraction of Sugawara (studied with the traditional method in [19]), and related constructions (GKO, Virasoro) soon.

In relation to the construction of WZW models, there are strong indications that the family of solvable Lie algebras introduced in [34] can be obtained through a graded contraction, although they cannot be obtained from a Inönü–Wigner contraction. For instance, a $\mathbb{Z}_3$ graded structure is inherent in these algebras. However, for the algebras $\mathcal{A}_{3m} (m \geqslant 3)$, the number of possible algebras to be contracted (which have the correct dimension) is huge, and there is no systematical way to identify them yet. This study is postponed to a future work.

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